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Computation of M-polynomial and Topological Indices of Hypergraphs with Linkages to Brain Functions

Anagha, P.[1] and Selvakumar, R.*[1]

[1]*Department of Mathematics, Vellore Institute of Technology, Vellore,Tamil Nadu, India*

E-mail: rselvakumar@vit.ac.in
**Corresponding author*

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Abstract

To study the functional characteristics of a network, the hypergraph model is more effective than the standard graph representation. M-polynomial is the most commonly used tool in theoretical chemistry to calculate accurate expressions for various degree-based topological indices. This study establishes the concept of M-polynomials for hypergraphs and derives the corresponding formulae for the topological indices as required. Furthermore, this study presents the linkages between hypergraph topological indices and the functions of the brain. Here, hypergraph topological indices are derived from a parameter that quantifies the number of multi-functioning regions involved in one function, thereby capturing the correlation between different functions.

Keywords: hypergraph; topological indices; M-polynomial; Brodmann area.

1 Introduction

Euler was the first to solve the Seven Bridges of Königsberg issue using a graph technique in 1736. Sylvester first used the term "graph" in 1878. Graphs are widely utilized in many different fields. Graphs have seen significant advancements in the first half of the 20th century. Still, research and applications in various domains, including computer science, physics, mathematics, biology, and social and economic network analysis, are ongoing. To expand on the graph technique, Berge presented hypergraphs [4]. Graphs are limited to supporting pairwise relationships. Hypergraphs are a logical way to describe collaborative networks and other scenarios since they maintain multiadic links.

The structure of hypergraphs significantly expands the idea of a graph. A hypergraph of a brain can be created by representing brain areas as vertices and brain activity as hyperedges. Unlike ordinary graphs, hypergraphs can describe intricate interactions between vertices beyond merely connections or edges [5, 25]. Modeling and analyzing brain graphs can be facilitated using hypergraphs since they can reflect intricate interactions between nodes, or different parts of the brain. A great way to utilize hypergraphs in human brain research is to analyze functional connectivity [36]. The connections between activity levels in different brain areas are referred to as functional connectivity [11].

Hypergraphs can help in capturing complicated functional connections since they allow multiple brain areas to be connected at once via a hyperedge, instead of merely pairing brain regions as in regular graphs [1]. Taking everything into account, hypergraphs are an effective tool for analyzing and modeling the complex connections between different brain areas. Chemistry has used hypergraph theory [24, 23]. Research in [23] shows that the hypergraph model accurately describes molecular structures, resulting in more diverse invariant behavior. Hypergraphs have been used to study many significant mathematical chemistry issues, such as spectrum features [26, 30] and various topological indices [35].

Many graph polynomials were brought into the literature, and some of them were useful in mathematical chemistry. The Hosoya polynomial is important in the domain of distance-based topological indices [19]. The Wiener index, in particular, may be derived as the first derivative of the Hosoya polynomial, evaluated at one, while the hyper-Wiener index [7] and Tratch-Stankevich-Zefirov index can be produced in the same way [6]. The Zhang-Zhang polynomial (also called the Clar covering polynomial) [9, 37], the Schultz polynomial [18], the Tutte polynomial [10], the matching polynomial [12, 15], and others are also chemically significant polynomials. The M-polynomial is an algebraic polynomial that finds significant application in the field of theoretical chemistry. It has a significant impact on the computation of precise expressions for several degree-based topological indices [8, 27].

Chemical graph theory introduces many topological indices, especially degree-based ones [20]. A new review [16] emphasizes the importance of degree-based indicators and their correlation with octane isomer physicochemical properties, as reported in [17]. Topological indices refer to quantitative representations of structural data of a graph. Considerable attention has been devoted to the topic of chemical graph theory, which represents a convergence of graph theory and chemistry within the realm of mathematics. Topological indices have demonstrated utility in the determination of quantitative structure-activity relationships (QSARs) and quantitative structure-property relationships (QSPRs) [22, 38]. Various topological indices are derived from diverse properties, such as degree, distance, eccentricity, and other factors [3, 31].

A plethora of graph invariants based on degrees has been extensively studied in the literature of both mathematics and mathematical chemistry. The first Zagreb index, symmetric division index, Randić index, second Zagreb index, harmonic index, modified second Zagreb index of a graph H are defined as:

$$M_1(H) = \sum_{rs \in E(H)} [d(r) + d(s)],$$

$$SDD(H) = \sum_{rs \in E(H)} \left[\frac{\min\{d(r), d(s)\}}{\max\{d(r), d(s)\}} + \frac{\max\{d(r), d(s)\}}{\min\{d(r), d(s)\}}\right], [14]$$

$$R(H) = \sum_{rs \in E(H)} \frac{1}{\sqrt{d(r) * d(s)}}, [2]$$

$$M_2(H) = \sum_{rs \in E(H)} [d(r) * d(s)],$$

$$H(H) = \sum_{rs \in E(H)} \frac{2}{d(r) + d(s)}, [39]$$

$$M_2^*(H) = \sum_{rs \in E(H)} \frac{1}{d(r) * d(s)}, [32]$$

respectively where $r, s \in V(H)$. Topological indices are numerical measures that effectively capture diverse connection aspects inherent in a network [29].

This work is grounded on using a novel parameter, hyperedge degree denoted as $d_h(c)$, and topological indices based on $d_h(c)$. To better resemble the edge degree of a normal graph, the newly introduced parameter $d_h(c)$ is defined to be equal to $\sum_{v \in c} d_h(v) - |c|$ where $|c|$ is the number of vertices in c. This parameter, $d_h(c)$, is used to evaluate the level of connectedness between a hyperedge and other hyperedges, and $d_h(v)$ denotes the number of hyperedges that a vertex v is member of.

The introduction serves as the initial segment of this work. The second portion defines M-polynomials for hypergraphs and novel topological indices for hypergraphs based on the parameter $d_h(c)$. Moreover, the derivation of the formulae corresponding to these topological indices is included. The final section describes the application of these hypergraph topological indices in brain studies.

2 Hypergraph Topological Indices

This section focuses on hypergraph topological indices based on $d_h(c)$ and comprises the definition of the M-polynomial for hypergraphs, as well as the derivation of formulae for various hypergraph topological indices because the M-polynomial holds notable significance within the context of degree-based topological indices. Some important notations supporting this section are as follows:

- $S_x(f(x, y)) = \int \frac{f(x, y)}{x} dx$,
- $S_y(f(x, y)) = \int \frac{f(x, y)}{y} dy$,

- $Jf(x,y) = f(x,x)$,
- $D_x(f(x,y)) = x\dfrac{\partial f(x,y)}{\partial x}$.

Definition 2.1. [33] *A pair (X, E) is said to be a hypergraph H if E is the hyperedge set, X is the vertex set, and every hyperedge is a non-empty subset of X.*

Definition 2.2. *The first Zagreb hyperedge index M_{H_1} of hypergraph H is defined as:*

$$M_{H_1} = \sum_{c\cap a=\phi} [d_h(c) + d_h(a)], \tag{1}$$

where c and a are hyperedges.

Definition 2.3. *The second Zagreb hyperedge index M_{H_2} of hypergraph H is defined as:*

$$M_{H_2} = \sum_{c\cap a=\phi} [d_h(c) \times d_h(a)], \tag{2}$$

where c and a are hyperedges.

Definition 2.4. *The modified second Zagreb hyperedge index $M_{H_2^*}$ of hypergraph H is defined as:*

$$M_{H_2^*} = \sum_{c\cap a=\phi} \frac{1}{d_h(c) \times d_h(a)}, \tag{3}$$

where c and a are hyperedges.

Definition 2.5. *The harmonic hyperedge index H_H of hypergraph H is defined as:*

$$H_H = \sum_{c\cap a=\phi} \frac{2}{d_h(c) + d_h(a)}, \tag{4}$$

where c and a are hyperedges.

Definition 2.6. *The general Randić hyperedge index R_H^α of hypergraph H is defined as:*

$$R_H^\alpha = \sum_{c\cap a=\phi} (d_h(c) \times d_h(a))^\alpha, \tag{5}$$

where c and a are hyperedges and $\alpha \in R$.

Definition 2.7. *The symmetric division hyperedge index SDD_H of hypergraph H is defined as:*

$$SDD_H = \sum_{c\cap a=\phi} \left(\frac{\min\{d_h(c), d_h(a)\}}{\max\{d_h(c), d_h(a)\}} + \frac{\max\{d_h(c), d_h(a)\}}{\min\{d_h(c), d_h(a)\}} \right), \tag{6}$$

where c and a are hyperedges.

Definition 2.8. *M-polynomial of a hypergraph H is*

$$M(H, x, y) = \sum_{c\cap a=\phi} x^{d_h(c)} y^{d_h(a)}, \tag{7}$$

where c and a are hyperedges.

Theorem 2.1. *Let H be a hypergraph, then the first Zagreb hyperedge index:*

$$M_{H_1} = \frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}\bigg|_{y=x=1}. \tag{8}$$

Proof. From Definition 2.8,

$$\begin{aligned}
M(H,x,y) &= \sum_{c\cap a=\phi} x^{d_h(c)} y^{d_h(a)},\\
\frac{\partial M}{\partial x} &= \sum_{c\cap a=\phi} d_h(c) x^{d_h(c)-1} y^{d_h(a)},\\
\frac{\partial M}{\partial y} &= \sum_{c\cap a=\phi} x^{d_h(c)} d_h(a) y^{d_h(a)-1},\\
\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} &= \sum_{c\cap a=\phi} (d_h(c) x^{d_h(c)-1} y^{d_h(a)} + x^{d_h(c)} d_h(a) y^{d_h(a)-1}),\\
\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}\bigg|_{y=x=1} &= \sum_{c\cap a=\phi} [d_h(c) + d_h(a)].
\end{aligned}$$

From (1), $M_{H_1} = \frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}\Big|_{y=x=1}$. □

Theorem 2.2. *Let H be a hypergraph, then the second Zagreb hyperedge index:*

$$M_{H_2} = \frac{\partial M}{\partial x} \cdot \frac{\partial M}{\partial y}\bigg|_{y=x=1}. \tag{9}$$

Proof. From Definition 2.8,

$$\begin{aligned}
M(H,x,y) &= \sum_{c\cap a=\phi} x^{d_h(c)} y^{d_h(a)},\\
\frac{\partial M}{\partial x} &= \sum_{c\cap a=\phi} d_h(c) x^{d_h(c)-1} y^{d_h(a)},\\
\frac{\partial M}{\partial y} &= \sum_{c\cap a=\phi} x^{d_h(c)} d_h(a) y^{d_h(a)-1},\\
\frac{\partial M}{\partial x} \cdot \frac{\partial M}{\partial y} &= \sum_{c\cap a=\phi} (d_h(c) x^{d_h(c)-1} y^{d_h(a)} \times x^{d_h(c)} d_h(a) y^{d_h(a)-1}),\\
\frac{\partial M}{\partial x} \cdot \frac{\partial M}{\partial y}\bigg|_{y=x=1} &= \sum_{c\cap a=\phi} [d_h(c) \times d_h(a)].
\end{aligned}$$

From (2), $M_{H_2} = \frac{\partial M}{\partial x} \cdot \frac{\partial M}{\partial y}\Big|_{y=x=1}$. □

Theorem 2.3. *Let H be a hypergraph, then modified second Zagreb hyperedge index:*

$$M_{H_2^*} = (S_x \cdot S_y) M(H,x,y)\bigg|_{y=x=1}. \tag{10}$$

Proof.

$$
\begin{aligned}
S_x &= \int \frac{M(H,x,y)}{x}dx \\
&= \sum_{c\cap a=\phi} \int (x^{d_h(c)-1}y^{d_h(a)})dx \\
&= \sum_{c\cap a=\phi} \frac{y^{d_h(a)}x^{d_h(c)}}{d_h(c)}, \\
S_y &= \int \frac{M(H,x,y)}{y}dy \\
&= \sum_{c\cap a=\phi} \int (x^{d_h(c)}y^{d_h(a)-1})dy \\
&= \sum_{c\cap a=\phi} \frac{x^{d_h(c)}y^{d_h(a)}}{d_h(a)}, \\
S_x\cdot S_y &= \sum_{c\cap a=\phi} \frac{x^{2d_h(c)}y^{2d_h(a)}}{d_h(c)d_h(a)}, \\
(S_x\cdot S_y)M(x,y)\Big|_{y=x=1} &= \sum_{c\cap a=\phi} \frac{1}{d_h(c)\times d_h(a)}.
\end{aligned}
$$

From (3), $M_{H_2^*} = (S_x\cdot S_y)M(H,x,y)\big|_{y=x=1}$. □

Theorem 2.4. *Let H be a hypergraph, then the harmonic hyperedge index:*

$$H_H = 2S_x(JM(H,x,y))\Big|_{x=1}. \tag{11}$$

Proof.

$$
\begin{aligned}
JM(H,x,y) = M(H,x,x) &= \sum_{c\cap a=\phi} x^{d_h(c)}x^{d_h(a)} \\
&= \sum_{c\cap a=\phi} x^{d_h(c)+d_h(a)}, \\
S_x(JM(H,x,y)) &= \int \frac{JM(H,x,y)}{x}dx \\
&= \sum_{c\cap a=\phi} \int x^{d_h(c)+d_h(a)-1}dx \\
&= \sum_{c\cap a=\phi} \frac{x^{d_h(c)+d_h(a)}}{d_h(c)+d_h(a)}, \\
2S_x(JM(H,x,y))\Big|_{x=1} &= \sum_{c\cap a=\phi} \frac{2}{d_h(c)+d_h(a)}.
\end{aligned}
$$

From (4), $H_H = 2S_x(JM(H,x,y))\big|_{x=1}$. □

Theorem 2.5. *Let H be a hypergraph, then general Randić hyperedge index R_H^α is*

$$\sum_{c\cap a=\phi} (d_h(c)\times d_h(a))^\alpha = (D_x^\alpha\cdot D_y^\alpha)M(H,x,y)\Big|_{y=x=1}. \tag{12}$$

Proof.

$$D_x = x\frac{\partial M}{\partial x} = \sum_{c\cap a=\phi} d_h(c)x^{d_h(c)}y^{d_h(a)},$$
$$D_y = y\frac{\partial M}{\partial y} = \sum_{c\cap a=\phi} x^{d_h(c)}d_h(a)y^{d_h(a)},$$
$$D_x^\alpha = \sum_{c\cap a=\phi} d_h(c)^\alpha(x^{d_h(c)}y^{d_h(a)}),$$
$$D_y^\alpha = \sum_{c\cap a=\phi} d_h(a)^\alpha(x^{d_h(c)}y^{d_h(a)}),$$
$$D_x^\alpha \cdot D_y^\alpha = \sum_{c\cap a=\phi} d_h(c)^\alpha d_h(a)^\alpha(x^{d_h(c)}y^{d_h(a)}),$$
$$D_x^\alpha \cdot D_y^\alpha\Big|_{y=x=1} = \sum_{c\cap a=\phi} d_h(c)^\alpha d_h(a)^\alpha.$$

From (5), $R_H^\alpha = D_x^\alpha \cdot D_y^\alpha\big|_{y=x=1}$. □

Theorem 2.6. *Let H be a hypergraph, then the symmetric division hyperedge index SDD_H is*

$$\sum_{c\cap a=\phi}\left(\frac{\min\{d_h(c),d_h(a)\}}{\max\{d_h(c),d_h(a)\}}+\frac{\max\{d_h(c),d_h(a)\}}{\min\{d_h(c),d_h(a)\}}\right) = (S_y\cdot D_x + S_x\cdot D_y)M(H,x,y)\Big|_{y=x=1}. \quad (13)$$

Proof.

$$D_x = \frac{\partial M}{\partial x} = \sum_{c\cap a=\phi} d_h(c)x^{d_h(c)-1}y^{d_h(a)},$$
$$D_y = \frac{\partial M}{\partial y} = \sum_{c\cap a=\phi} x^{d_h(c)}d_h(a)y^{d_h(a)-1},$$
$$\begin{aligned} S_x &= \int \frac{M(H,x,y)}{x}dx \\ &= \sum_{c\cap a=\phi}\int (x^{d_h(c)-1}y^{d_h(a)})dx \\ &= \sum_{c\cap a=\phi}\frac{y^{d_h(a)}x^{d_h(c)}}{d_h(c)}, \\ S_y &= \int \frac{M(H,x,y)}{y}dy \\ &= \sum_{c\cap a=\phi}\int (x^{d_h(c)}y^{d_h(a)-1})dy \\ &= \sum_{c\cap a=\phi}\frac{x^{d_h(c)}y^{d_h(a)}}{d_h(a)}, \end{aligned}$$

$$S_x \cdot D_y = \sum_{c \cap a = \phi} \frac{d_h(a)}{d_h(c)} x^{2d_h(c)} y^{2d_h(a)-1},$$

$$S_y \cdot D_x = \sum_{c \cap a = \phi} \frac{d_h(c)}{d_h(a)} x^{2d_h(c)-1} y^{2d_h(a)},$$

$$(S_y \cdot D_x + S_x \cdot D_y) = \sum_{c \cap a = \phi} \left(\frac{d_h(a)}{d_h(c)} x^{2d_h(c)} y^{2d_h(a)-1} + \frac{d_h(c)}{d_h(a)} x^{2d_h(c)-1} y^{2d_h(a)}\right),$$

$$(S_y \cdot D_x + S_x \cdot D_y) M(H,x,y)\Big|_{y=x=1} = \sum_{c \cap a = \phi} \left(\frac{d_h(a)}{d_h(c)} + \frac{d_h(c)}{d_h(a)}\right).$$

From (6), $SDD_H = (S_y \cdot D_x + S_x \cdot D_y).M(H,x,y)\big|_{y=x=1}$. □

3 Role of Hypergraph Topological Indices in Alzheimer's Disease

A brain is represented by a hypergraph, whose vertices are brain regions and hyperedges are brain functions. Hypergraphs are useful for modeling and analyzing brain graphs because they may show complex relationships between various brain regions. Since hypergraphs enable many brain areas to be connected simultaneously by a hyperedge rather than connecting two brain areas as in normal graphs [1], they can aid in the capture of intricate functional linkages. If everything is taken into account, hypergraphs are a useful tool for representing and examining the complex interactions between various brain areas.

The incidence matrix of the brain hypergraph is given in Table 1. Here in the brain hypergraph, each Brodmann area represents a vertex, and brain functions such as sensing/touching, motor movement, taste, visual, smell, auditory, memory, speech, emotions, attention, navigation, and executive function represent hyperedges $c_1, c_2, \ldots, c_{12}$ respectively. The Brodmann areas of the cerebral cortex are based on the types of neurons and synapses typically found in each area. The 52 Brodmann regions consist of the primary auditory cortex, the primary visual cortex, the primary motor cortex, the primary somatosensory cortex, the premotor cortex, the somatosensory association cortex, the supplementary motor cortex, the anterior prefrontal cortex, the dorsolateral/anterior prefrontal cortex, and the primary visual and auditory cortices [21]. Brodmann areas and their functions are utilized from https://www.simplypsychology.org/brodmann-areas.html and https://www.kenhub.com/en/library/anatomy/brodmann-areas for the construction of the brain hypergraph.

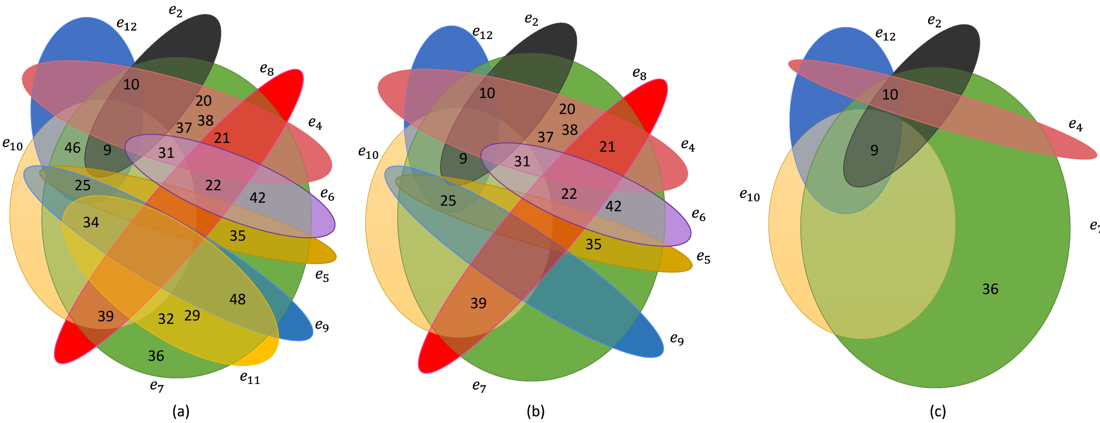
Table 1: Incidence matrix of brain hypergraph.

	1,2,3	4	5	6	7	8	9	10	11	12	13–16	17–19	20	21	22	23–24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44–45	46	47	48	52
c_1	1	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
c_2	0	1	0	1	0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
c_3	0	0	1	0	1	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
c_4	0	0	1	0	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
c_5	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
c_6	0	0	1	1	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
c_7	0	0	0	0	0	0	1	1	0	0	0	0	1	1	1	0	1	0	0	0	1	0	1	1	0	1	1	1	1	1	1	0	0	1	0	0	1	0	1	0
c_8	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0	1	0	0
c_9	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	1	0	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0
c_{10}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1
c_{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0
c_{12}	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$d_h(v)$	1	1	5	2	5	2	4	4	3	2	8	1	2	3	3	1	4	1	1	1	2	2	4	2	1	4	2	1	2	2	3	2	1	2	1	2	3	2	3	1

Define the new hypergraph parameter, hyperedge degree $d_h(c)$ as $d_h(c) = \sum_{\forall v \in c} d_h(v) - |c|$, where c is a hyperedge and $d_h(v)$ is the number of hyperedges to which v belongs. Using the definition of $d_h(c)$ and the data in Table 1, $d_h(c_1), d_h(c_2), \ldots, d_h(c_{12})$ were calculated, and they are $37, 38, 39, 51, 40, 43, 32, 38, 13, 13, 7, 39$ respectively. So, the M-polynomial of the brain hypergraph is

$$\begin{aligned} M(H,x,y) &= x^{d_h(c_1)}[y^{d_h(c_7)} + y^{d_h(c_{10})} + y^{d_h(c_{11})}] + x^{d_h(c_2)}[y^{d_h(c_3)} + y^{d_h(c_5)} + y^{d_h(c_9)} + y^{d_h(c_{11})}] \\ &\quad + x^{d_h(c_3)}[y^{d_h(c_7)} + y^{d_h(c_{10})} + y^{d_h(c_{11})}] + x^{d_h(c_4)}y^{d_h(c_{11})} + x^{d_h(c_5)}[y^{d_h(c_{10})} + y^{d_h(c_{11})}] \\ &\quad + x^{d_h(c_6)}[y^{d_h(c_9)} + y^{d_h(c_{11})}] + x^{d_h(c_8)}y^{d_h(c_{11})} \\ &= x^{37}[y^{32} + y^{13} + y^7] + x^{38}[y^{39} + y^{40} + y^{13} + y^7] + x^{39}[y^{32} + y^{13} + y^7] + x^{51}y^7 \\ &\quad + x^{40}[y^{13} + y^7] + x^{43}[y^{13} + y^7] + x^{38}y^7. \end{aligned}$$

This $d_h(c)$ value indicates how much this edge c is interconnected with or collaborates with other edges. For example, here $d_h(c_4)$ (c_4 contains regions connected with visual processing) is high. This means that regions involved in visual function have connections with other functions. Since $\max\{d_h(v); v \in c_4\} = 8$, visual regions are interconnected with at least 7 functions and a maximum of 12. So, visual function will be impacted if there are problems with other functions, like an early symptom or aftereffect.



(a) Normal person. (b) Mild Alzheimer's. (c) Moderate Alzheimer's.

Figure 1: Brain hypergraph focused on the memory regions of 1(a), 1(b) and 1(c).

Alzheimer's patients experience greater memory loss. It is a memory-related disease. The brain hypergraph in Figure 1 is constructed using the data released by National Institute of Health (NIH).

Theorem 3.1. *Let G be a hypergraph and $H = G - v$ be a sub-hypergraph of G where v is a vertex of G with $d_h(v) > 1$ and removal of v doesnot disconnect G then $M_{H_1}(H) < M_{H_1}(G)$ and $M_{H_2}(H) < M_{H_2}(G)$.*

Proof. Let c be a hyperedge and $d_h(c) = \sum_{v \in c} d_h(v) - |c|$. So, $d_h(v) > 1$ means there exist at least two hyperedges c and ϵ such that $v \in c \cap \epsilon$. Therefore, $d_h(c) > d_h(c - v)$ and $d_h(\epsilon) > d_h(\epsilon - v)$. It implies $M_{H_1}(H) < M_{H_1}(G)$ and $M_{H_2}(H) < M_{H_2}(G)$. □

Theorem 3.2. *Let G be a hypergraph and $H = G - v$ be a sub-hypergraph of G where v is a vertex of G with $d_h(v) > 1$, then $M_{H_2^*}(H) > M_{H_2^*}(G)$ and $H_H(H) > H_H(G)$.*

Proof. $d_h(v) > 1$ means there exist at least two hyperedges c and ϵ such that $v \in c \cap \epsilon$. Therefore, $d_h(c) > d_h(c - v) \implies \dfrac{1}{d_h(c)} < \dfrac{1}{d_h(c - v)}$ and $d_h(\epsilon) > d_h(\epsilon - v) \implies \dfrac{1}{d_h(\epsilon)} < \dfrac{1}{d_h(\epsilon - v)}$. It implies $M_{H_2^*}(H) > M_{H_2^*}(G)$ and $H_H(H) > H_H(G)$. □

Theorem 3.3. *Let G be a hypergraph and N be the collection of vertices whose removal disconnects G. If $d_h(c) \neq 0$ for any hyperedge c of $G - N$, then,*

$$M_{H_1}(G-N) > M_{H_1}(G), \quad M_{H_2}(G-N) > M_{H_2}(G), \quad M_{H_2^*}(G-N) > M_{H_2^*}(G),$$
$$H_H(G-N) > H_H(G), \quad R_H^\alpha(G-N) > R_H^\alpha(G), \quad SDD_H(G-N) > SDD_H(G).$$

Proof. Let $H = G - N$. Since N is removed, the hyperedges c and ϵ will become disjoint. So, the number of hyperedge pairs (c, a) such that $c \cap a = \phi$ will increase. Therefore, since $d_h(\epsilon) \geq 1$ $\forall \epsilon \in E(G - N)$, the sum is gained. Hence,

$$M_{H_1}(H) > M_{H_1}(G), \quad M_{H_2}(H) > M_{H_2}(G), \quad M_{H_2^*}(H) > M_{H_2^*}(G),$$
$$H_H(H) > H_H(G), \quad R_H^\alpha(H) > R_H^\alpha(G), \quad SDD_H(H) > SDD_H(G).$$

□

The human brain serves as the central hub for the nervous system, facilitating various cognitive processes such as thoughts, memory, movement, and emotions [34]. Alzheimer's disease is widely recognized as the primary form of dementia, characterized by a progressive deterioration of several cognitive domains. In conjunction with physical infirmity, aphasia, gait and balance impairments, and cerebrovascular illnesses can potentially contribute towards cognitive impairment and dementia [13, 28]. Unfortunately, these conditions often go unnoticed or receive insufficient attention from both those affected and medical practitioners.

Due to the complexity of Alzheimer's disease, it is unlikely that a single drug or other treatment will ever be able to cure all patients. Nonetheless, experts have made significant strides in recent years. This section explains how this hypergraph concept can be implemented in investigations of brain disorders, particularly Alzheimer's.

Every brain region is a vertex in the brain hypergraph, and brain functions are hyperedges. Every stage of Alzheimer's affects a few functions. This is a result of some brain regions being damaged. The severity of the ensuing symptoms varies depending on the affected region. During the final phases of Alzheimer's disease, certain brain functions become fully impaired. Thus, it depicts the disconnection of the brain's hypergraph. Certain brain regions are injured during certain periods of Alzheimer's disease, which affects how certain functions intersect. Thus, it might disintegrate brain processes.

The topological indices of the brain hypergraph indicate the effects of damage to each major brain region. Theorem 3.2 and Theorem 3.3 show that the topological indices $M_{H_2^*}$ and H_H increase with each stage of Alzheimer's. Therefore, it can be used to identify precisely the stage of Alzheimer's using the threshold value. Theorem 3.1 states that the topological indices M_{H_1} and M_{H_2} values will decrease if the damage to brain regions does not cause the brain hypergraph to become disconnected. However, Theorem 3.3 states that if it begins to disconnect the graph, the topological indices M_{H_1} and M_{H_2} will suddenly rise. Situations with zero hyperedge degree are ignored.

4 Conclusion

The newly introduced parameter $d_h(c)$ is accounting the occurrence of intersections or partnerships. This implies that the value of the parameter visualizes the presence and quantity of shared components among hyperedges, given that hyperedges are composed of several vertices. Utilizing this parameter and the associated indices will be very beneficial in circumstances where partnerships or intersections are important.

In the context of this topic, it is observed that $M_{H_2^*}$ and H_H exhibit an upward trend with the progression of Alzheimer's disease. The absence of hyperedge intersections has an impact on the indices, specifically affecting the indices M_{H_1} and M_{H_2}. In the severe stage of Alzheimer's disease, the functional linkages associated with memory become impaired, resulting in a rise in the values of the indices M_{H_1} and M_{H_2}. A discrepancy is observed in the indices' values throughout the severe stages of Alzheimer's disease.

The process of brain parcellation facilitates the comprehension of the structural and functional arrangement of the cerebral cortex. The utilization of functional MRI and connectivity analysis is valuable in the characterization of individual brain parcels in vivo. This study has the potential for expansion through the utilization of brain parcellation techniques, which involve the segmentation of the brain into distinct regions that have certain functions, as well as the incorporation of machine learning methodologies. The rise in the second modified Zagreb and harmonic indices at each stage of Alzheimer's disease allows for the determination of a threshold value for each phase, using the aforementioned indices. This can assist in the detection and classification of the various phases of Alzheimer's disease.

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